

## ON THE IMAGE OF THE TOTALING FUNCTOR

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**ABSTRACT.** Let  $A$  be a polynomial ring in  $d$  variables over a field. In this paper, we investigate the image of the totaling functor between the derived category of graded  $A$ -modules and the derived category of DG  $A$ -modules. In particular, we show that when  $d \geq 2$ , there exist semifree DG modules of rank at least four which are not quasiisomorphic to the totaling of a complex of graded  $A$ -modules. However, when  $d = 1$  we show that the totaling functor is onto. Moreover, when  $A$  is an arbitrary DG algebra with trivial differential, we exhibit a special class of semifree DG modules over  $A$  which are always equal to the totaling of some complex of graded  $A$ -modules.

## INTRODUCTION

The main subjects in the study of differential graded homological algebra are DG algebras and the DG modules over them. These structures can be viewed as algebras and their modules, each possessing a differential that ‘cooperates’ with the action of the algebra. Such differential graded objects arise naturally, for example, in the form of Koszul complexes and certain singular cohomology rings. Furthermore, the literature abounds with conditions for their existence as (minimal) projective resolutions [3–7].

Our interest in differential graded homological algebra lies in understanding the richness of the derived category  $\mathbf{DDG}(A)$  of DG modules over a fixed DG algebra  $A$ . This category can be thought of as a localization of the homotopy category of DG  $A$ -modules. Specifically, we focus on the relation between  $\mathbf{DG}(A)$  and the category  $\mathbf{ChGr}(A)$  of complexes of graded modules over the underlying graded algebra of  $A$ . When  $A$  is arbitrary, it is not difficult to see that the objects of  $\mathbf{DG}(A)$  have a more complicated structure than those of  $\mathbf{ChGr}(A)$ . However, over a DG algebra with trivial differential, the same observation is not immediate.

In order to compare the categories over such a DG algebra, we study the totaling functor between them,  $\mathrm{Tot} : \mathbf{ChGr}(A) \rightarrow \mathbf{DG}(A)$ . One nice characteristic of this functor is its preservation of quasiisomorphisms—that is, chain maps which induce isomorphism in homology. Such chain maps define the concept of isomorphism in the derived category; we can thus extend the totaling functor.

$$\mathrm{Tot} : \mathbf{DGr}(A) \rightarrow \mathbf{DDG}(A)$$

In this work, we investigate the image of the totaling functor in the case that  $A$  is a polynomial ring over a field.

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The paper is organized as follows. In Section 1, we define the preliminary concepts and categories. Section 2, which contains our results, is outlined in three subsections. In Section 2.1, we study the image of Tot over a polynomial ring in at least two variables. These results naturally motivate the subject of Section 2.2, wherein we define, for an arbitrary DG algebra  $A$  with trivial differential, a special class of DG modules which lies in the image of Tot. Finally, in Section 2.3 we investigate the image of Tot over a polynomial ring in one variable.

## 1. BACKGROUND

Throughout, let  $k$  denote a field and  $R$  a commutative ring with identity. Unless otherwise noted, all modules are assumed to be left modules. Furthermore, when  $\mathbb{N}$  is used as an indexing set, it shall contain zero.

**1.1. Complexes.** A *chain complex* of  $R$ -modules given by

$$\cdots \rightarrow X_{i+2} \xrightarrow{\partial_{i+2}^X} X_{i+1} \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \xrightarrow{\partial_{i-1}^X} X_{i-2} \rightarrow \cdots$$

shall be denoted  $X = (X^\natural, \partial^X)$ , where  $X^\natural$  represents the underlying graded  $R$ -module of the complex, or simply  $(X^\natural, \partial)$  when there is no risk of confusion. With the convention that  $X^d = X_{-d}$ , we define similarly the *cochain complex*  $X$ , given by:

$$\cdots \rightarrow X^{i-2} \xrightarrow{\partial_X^{i-2}} X^{i-1} \xrightarrow{\partial_X^{i-1}} X^i \xrightarrow{\partial_X^i} X^{i+1} \xrightarrow{\partial_X^{i+1}} X^{i+2} \rightarrow \cdots$$

The notions of chain and cochain complexes are one and the same; the definitions which follow will be stated in terms of chain complexes. As per clarity, we shall utilize both types of complexes in the sequel.

For a complex  $X$ , we use  $Z(X)$  and  $B(X)$  to denote the graded  $R$ -modules of *cycles* and *boundaries*, respectively, of  $X$ . They are defined as follows.

$$Z_i(X) = \ker \partial_i^X$$

$$B_i(X) = \operatorname{im} \partial_{i+1}^X$$

Hence the graded *homology* module of  $X$  is given by  $H(X) = \bigoplus_{i \in \mathbb{Z}} H_i(X)$ , where

$$H_i(X) = Z_i(X) / B_i(X).$$

If  $H_i(X) = 0$  for all  $|i| \gg 0$ , we say that  $X$  is *homologically bounded*.

For a given complex  $X$  of  $R$ -modules, the  $d^{\text{th}}$  *shift* of  $X$  is the complex  $\Sigma^d X$  of  $R$ -modules whose underlying graded module is given by

$$(\Sigma^d X)_i = X_{i-d}$$

for each  $i \in \mathbb{Z}$ . Furthermore, for every  $x \in X$ , we denote by  $\sigma^d(x)$  its image in  $\Sigma^d X$ . With this notation, we now define the differential on the complex  $\Sigma^d X$  to be given by

$$\partial_i^{\Sigma^d X}(\sigma^d x) = (-1)^d \sigma^d(\partial_{i-d}^X(x))$$

for each  $\sigma^d x \in (\Sigma^d X)_i$  and all  $i \in \mathbb{Z}$ . For simplicity, we set  $\Sigma X = \Sigma^1 X$ .

A homomorphism  $\mu : X \rightarrow Y$  of complexes of  $R$ -modules is called a *chain map* provided that

$$\partial^Y \circ \mu = (-1)^{|\mu|} \mu \circ \partial^X$$

where  $|\mu|$  denotes the degree of  $\mu$ . Moreover,  $\mu$  is a *morphism* of complexes if  $|\mu| = 0$ .

*Remarks 1.1.1.* (1) The map  $\sigma^d : X \rightarrow \Sigma^d X$  is a bijective chain map of complexes; it is called the *suspension map*.

(2) If  $X$  has a trivial differential, it represents a graded  $R$ -module; in this case we define similarly the  $d^{\text{th}}$  shift  $\Sigma^d X$  of  $X$  and the  $d^{\text{th}}$  suspension map  $\sigma^d : X \rightarrow \Sigma^d X$  on  $X$ .

**Definition 1.1.2.** Given two morphisms  $\mu, \lambda : X \rightarrow Y$  of complexes of  $R$ -modules, a *homotopy* between  $\mu$  and  $\lambda$  is a degree +1 homomorphism of complexes  $\sigma : X \rightarrow Y$  such that

$$\mu - \lambda = \partial^Y \circ \sigma + \sigma \circ \partial^X.$$

This condition is illustrated by the following diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X_{i+1} & \xrightarrow{\partial_{i+1}^X} & X_i & \xrightarrow{\partial_i^X} & X_{i-1} \longrightarrow \cdots \\ & & \downarrow \mu_{i+1} & \swarrow \sigma_i & \downarrow \mu_i & \swarrow \sigma_{i-1} & \downarrow \mu_{i-1} \\ & & Y_{i+1} & \xrightarrow{\partial_{i+1}^Y} & Y_i & \xrightarrow{\partial_i^Y} & Y_{i-1} \longrightarrow \cdots \end{array}$$

(Note: In the original image, the vertical arrows are labeled  $\mu_{i+1}, \lambda_{i+1}$  and  $\mu_i, \lambda_i$  respectively, with dashed arrows labeled  $\sigma_i$  and  $\sigma_{i-1}$  connecting them.)

In the case that such a  $\sigma$  exists, we say that  $\mu$  and  $\lambda$  are *homotopic*. A chain map  $\mu : X \rightarrow Y$  is said to be *null homotopic* if it is homotopic to the zero map.

**Fact 1.1.3.** If  $\mu : X \rightarrow Y$  is a morphism of complexes of  $R$ -modules, then  $\mu$  induces a degree zero homomorphism of graded  $R$ -modules

$$H(\mu) : H(X) \rightarrow H(Y).$$

Moreover, if  $\lambda : X \rightarrow Y$  is a morphism which is homotopic to  $\mu$ , then  $H(\mu) = H(\lambda)$ .

**Definition 1.1.4.** A morphism  $\mu : X \rightarrow Y$  of complexes of  $R$ -modules is called a *quasiisomorphism* if  $H(\mu) : H(X) \rightarrow H(Y)$  is an isomorphism of graded  $R$ -modules. Furthermore  $X$  and  $Y$  are said to be *quasiisomorphic*, denoted  $X \simeq Y$ , if there exists a sequence of chain maps linking  $X$  and  $Y$ , each of which is a quasiisomorphism.

*Remark 1.1.5.* It is important to note that if  $X$  and  $Y$  are complexes of  $R$ -modules such that  $H(X) \cong H(Y)$ , it is not necessarily true that  $X \simeq Y$ .

**1.2. Differential graded algebra.** In this section, all tensor products are assumed to be taken over  $R$ .

A complex  $A$  of  $R$ -modules is called a *differential graded algebra* (or *DG algebra*) over  $R$  if there exist morphisms

$$\begin{aligned} \mu^A : A \otimes A &\rightarrow A \\ \eta^A : R &\rightarrow A \end{aligned}$$

called the *product* and *structure map*, respectively, such that the following diagrams commute.

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{A \otimes \mu^A} & A \otimes A \\ \downarrow \mu^A \otimes A & & \downarrow \mu^A \\ A \otimes A & \xrightarrow{\mu^A} & A \end{array} \quad \begin{array}{ccc} & A \otimes A & \\ \eta^A \otimes A \nearrow & \downarrow \mu^A & \nwarrow A \otimes \eta^A \\ R \otimes A & \xlongequal{\quad} & A \xlongequal{\quad} A \otimes R \end{array}$$

The maps  $\mu^A$  and  $\eta^A$  are usually suppressed from notation, in the following sense: the product  $\mu^A(a \otimes a')$  is denoted  $aa'$ , and  $\eta^A(1)$  is denoted 1 and is called the *unit* of  $A$ . Furthermore, as the product is a morphism of complexes, it must satisfy the Leibniz rule. That is,

$$\partial(aa') = \partial(a)a' + (-1)^{|a|}a\partial(a')$$

for each  $a, a' \in A$ . A morphism  $\varphi : A \rightarrow B$  of complexes is moreover a *morphism* of DG algebras if  $\varphi(aa') = \varphi(a)\varphi(a')$  and  $\varphi(1) = 1$ .

*Remark 1.2.1.* If  $A = A^\natural$  has trivial differential, then  $A$  is merely a graded  $R$ -algebra.

A complex  $M$  of  $R$ -modules is called a *differential graded module* (or *DG module*) over  $A$  if there exists a morphism

$$\mu^{AM} : A \otimes M \rightarrow M$$

called the *action* of  $A$ , such that the following diagrams commute.

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{A \otimes \mu^{AM}} & A \otimes M \\ \mu^A \otimes M \downarrow & & \downarrow \mu^{AM} \\ A \otimes M & \xrightarrow{\mu^{AM}} & M \end{array} \quad \begin{array}{ccc} & & A \otimes M \\ & \nearrow \eta^A \otimes M & \downarrow \mu^{AM} \\ R \otimes M & \xlongequal{\quad} & M \end{array}$$

As before, we denote  $\mu^{AM}(a \otimes m)$  by  $am$ , and we require that for each  $a \in A$  and  $m \in M$ , the Leibniz rule is satisfied:

$$\partial^M(am) = \partial^A(a)m + (-1)^{|a|}a\partial^M(m)$$

A subcomplex  $N \subseteq M$  is called a *DG submodule* if  $\mu^{AM}(A \otimes N) \subseteq N$ .

*Remarks 1.2.2.* (1) Notice that the product in an arbitrary DG algebra  $A$  is not necessarily commutative. Therefore, unless otherwise stated, a DG  $A$ -module  $M$  shall be assumed to have a left  $A$ -module structure.

(2) All of the definitions and results of Section 1.1 can be analogously stated for a DG module  $M$  when thought of as a complex over  $R$ .

(3) If  $A = A^\natural$  is a DG algebra with trivial differential, then it is merely a graded algebra. In this case, the action of  $A$  on any DG  $A$ -module is that of a chain map of degree  $|a|$ .

**1.3. Semifreeness.** In this section, let  $A$  denote a DG algebra and  $M$  denote a DG  $A$ -module. A subset  $E \subseteq M^\natural$  is called a *semibasis* of  $M$  if

- (i)  $E$  is a basis of  $M^\natural$  over  $A^\natural$ , and
- (ii)  $E = \bigsqcup_{d \in \mathbb{N}} E_d$ , as a disjoint union, such that

$$\partial(E_d) \subseteq A \left( \bigsqcup_{i < d} E_i \right)$$

for all  $d \in \mathbb{N}$ .

A DG module that possesses such a semibasis is said to be *semifree*.

**Proposition 1.3.1.** [1, 8.2.3] *Let  $A$  be a DG algebra and  $M$  be a DG  $A$ -module. The following are equivalent.*

- (1)  $M$  is semifree.
- (2)  $M^\natural$  has a well-ordered basis  $E$  over  $A^\natural$  such that for each  $e \in E$

$$\partial(e) \in A(\{e' \in E \mid e' < e\}).$$

*Proof.* (1)  $\Rightarrow$  (2) Let  $E = \bigsqcup_{d \in \mathbb{N}} E_d$  be a semibasis for  $M$  over  $A$ . For each  $d \in \mathbb{N}$ , impose an ordering on  $E_d$ , and further suppose that whenever  $d' < d$ ,  $e' < e$  for every  $e' \in E_{d'}$  and  $e \in E_d$ . This implies that  $E$  has an ordering with the desired property.

(2)  $\Rightarrow$  (1) Suppose that  $E$  is a well-ordered basis for  $M^\natural$  over  $A^\natural$  such that  $\partial(e) \in A(\{e' \in E \mid e' < e\})$  for every  $e \in E$ . Set  $E_{-1} = \emptyset$  and  $M_{-1} = \{0\}$ , and for each  $d \in \mathbb{N}$ , recursively define  $E_d$  and  $M_d$  in the following manner:

$$E_d = \{e \in E \setminus \bigcup_{i < d} E_i \mid \partial(e) \in M_{d-1}\}$$

$$M_d = A \left( \bigcup_{i \leq d} E_i \right)$$

By this construction, the  $E_d$  are mutually disjoint. Furthermore, the well-ordering of  $E$  implies that  $E = \bigsqcup_{d \in \mathbb{N}} E_d$ , and the result follows.  $\square$

**Definition 1.3.2.** Let  $M$  be a DG module over a DG algebra  $A$ . A *semifree resolution* of  $M$  is a quasiisomorphism  $\pi : F \rightarrow M$  of DG  $A$ -modules, where  $F$  is semifree.

The following is a result of Avramov, Foxby, and Halperin [1, 8.3.3 and 8.3.4] concerning the abundance of semifree modules. For the reader's convenience, we shall sketch their proof below.

**Theorem 1.3.3.** *Every DG module has a semifree resolution.*

*Proof.* Let  $M$  be a DG  $A$ -module. Our goal is to construct, by induction on  $d \in \mathbb{N}$ , a sequence of inclusions of DG  $A$ -modules  $F^{d-1} \subseteq F^d$  and of morphisms  $\varepsilon^d : F^d \rightarrow M$  of DG  $A$ -modules such that  $\varepsilon^d|_{F^{d-1}} = \varepsilon^{d-1}$ . With this accomplished, we will demonstrate that  $F = \bigcup_{d \in \mathbb{N}} F^d$  is semifree and that  $\varepsilon = \varinjlim \varepsilon^d : F \rightarrow M$  is a quasiisomorphism.

First we consider the case when  $d = 0$ . Let  $Z^0 \subseteq M$  consist of cycles, chosen such that  $\text{cls}(Z^0)$  generates  $H(M)$  over  $H(A)$ . Further, let

$$E^0 = \{e_z \mid |e_z| = |z| \text{ for } z \in Z^0\}$$

be linearly independent over  $A^\natural$ , and define the DG  $A$ -module  $F^0$ , whose underlying graded module is given by  $(F^0)^\natural = A^\natural E^0$ , and whose differential is given by

$$\partial^{F^0} \left( \sum_{z \in Z^0} a_z e_z \right) = \sum_{z \in Z^0} \partial^A(a_z) e_z.$$

Furthermore, we note that

$$\varepsilon^0 \left( \sum_{z \in Z^0} a_z e_z \right) = \sum_{z \in Z^0} a_z z$$

defines a morphism  $L^0 \rightarrow M$  of DG  $A$ -modules.

Now, suppose that  $d > 0$  and that  $F^{d-1}$  and  $\varepsilon^{d-1} : F^{d-1} \rightarrow M$  have been defined. Let  $Z^d \subseteq F^{d-1}$  consist of cycles such that  $\text{cls}(Z^d)$  generates  $\ker H(\varepsilon^{d-1})$  over  $H(A)$ . Now let

$$E^d = \{e_z \mid |e_z| = |z| + 1 \text{ for } z \in Z^d\}$$

be linearly independent over  $A^\natural$ , and define the DG  $A$ -module  $F^d$  by

$$(F^d)^\natural = A^\natural E^d \oplus (F^{d-1})^\natural$$

and

$$\partial^{F^d} \left( \sum_{z \in Z^d} a_z e_z + f \right) = \sum_{z \in Z^d} \partial^A(a_z) e_z + \sum_{z \in Z^d} (-1)^{|a_z|} a_z z + \partial^{F^{d-1}}(f).$$

It is easily seen that  $F^{d-1}$ , when identified with its image in  $F^d$ , is a DG submodule of  $F^d$ . Notice that by construction, for any  $z \in Z^d$  there exists  $x_z \in M$  such that  $\varepsilon^{d-1}(z) = \partial(x_z)$ . Therefore, define

$$\varepsilon^d \left( \sum_{z \in Z^d} a_z e_z + f \right) = \sum_{z \in Z^d} a_z x_z + \varepsilon^{d-1}(f).$$

One can check that  $\varepsilon^d : F^d \rightarrow M$  defines a morphism of DG  $A$ -modules, and that  $\varepsilon^d|_{F^{d-1}} = \varepsilon^{d-1}$ . Therefore  $F = \bigcup_{d \in \mathbb{N}} F^d$  is a semifree DG  $A$ -module with semibasis  $E = \bigsqcup_{d \in \mathbb{N}} E^d$ . We will now show that the morphism  $\varepsilon = \varinjlim \varepsilon^d : F \rightarrow M$  is a quasiisomorphism of DG  $A$ -modules.

To this end, notice that, for every  $d \in \mathbb{N}$ , we have the following commutative diagram of  $A^\natural$ -linear homomorphisms of graded  $A^\natural$ -modules.

$$\begin{array}{ccccc} H(F^0) & \xrightarrow{\quad} & H(F^d) & \xrightarrow{\quad} & H(F) \\ & \searrow H(\varepsilon^0) & \downarrow H(\varepsilon^d) & \swarrow H(\varepsilon) & \\ & & H(M) & & \end{array}$$

By the way we constructed  $\varepsilon^0$ , it follows that  $H(\varepsilon^0)$  is surjective. This implies that  $H(\varepsilon^d)$  and  $H(\varepsilon)$  are surjective as well. To see that  $H(\varepsilon)$  is also injective, choose a cycle  $z' \in F$  such that  $\text{cls}(z') \in \ker H(\varepsilon)$ , and let  $d' \in \mathbb{N}$  be such that  $z' \in F^{d'}$  and  $\text{cls}(z') \in \ker H(\varepsilon^{d'})$ . We are guaranteed that such a  $d'$  exists by virtue of the fact that  $H(\varepsilon) = H(\varinjlim \varepsilon^d) = \varinjlim H(\varepsilon^d)$ , cf. [1, Theorem 4.6.9]. Now since we chose  $Z^{d'+1}$  so that  $\text{cls}(Z^{d'+1})$  generates the  $H(A)$ -module  $\ker H(\varepsilon^{d'})$ , we can write

$$z' = \sum_{z \in Z^{d'+1}} \zeta_z z + \partial^{F^{d'}}(f)$$

for some cycles  $\zeta_z \in H(A)$ , only finitely many of which are nonzero, and for some  $f \in F^{d'}$ . By construction, given any  $z \in Z^{d'+1}$ , there exists an  $e_z \in F^{d'+1}$  such that  $z = \partial^{F^{d'+1}}(e_z)$ . Thus,

$$z' = \partial^{F^{d'+1}} \left( \sum_{z \in Z^{d'}} (-1)^{|\zeta_z|} \zeta_z e_z + f \right)$$

which is contained in  $F^{d'+1} \subseteq F$ . Since  $\text{cls}(z') = 0$  in  $H(F)$ , it follows that  $H(\varepsilon)$  is injective, and thus  $\varepsilon$  is a quasiisomorphism.  $\square$

This result forms the differential graded analog of the abundance of projective modules, therefore making possible the study of differential graded homological algebra.

**1.4. The Categories.** For an arbitrary abelian category  $\mathcal{A}$ , the derived category  $\mathbf{DA}$  is constructed in three steps. First, we form the category  $\mathbf{Ch}\mathcal{A}$  of complexes over  $\mathcal{A}$  and the degree zero ‘chain maps’ between them. Next we pass to the homotopy category  $\mathbf{K}\mathcal{A}$  wherein homotopic morphisms in  $\mathbf{Ch}\mathcal{A}$  are equated. Finally, through a localization of  $\mathbf{K}\mathcal{A}$  which formally inverts quasiisomorphisms, we arrive at the derived category  $\mathbf{DA}$  (for details, see [8, Chapter 10]).

We are now equipped to define the categories which we study. To this end, fix an arbitrary DG algebra  $A$ . The DG  $A$ -modules, along with the degree zero chain maps between them, are the objects and morphisms, respectively, of the category  $\mathbf{DG}(A)$ . Because of the fact that any DG  $A$ -module is *a priori* a complex, forming the category  $\mathbf{ChDG}(A)$  en route to the derived category is unnecessary. For this reason, and in lieu of Theorem 1.3.3, the derived category  $\mathbf{DDG}(A)$  can be viewed as the category whose objects are semifree DG  $A$ -modules, and whose morphisms are homotopy classes of chain maps between such DG modules.

On the other hand, notice that an arbitrary DG algebra  $A$  can be viewed in terms of the underlying graded algebra  $A^\natural$ . With this in mind, the graded  $A^\natural$ -modules are precisely the objects of the category  $\mathbf{Gr}(A^\natural)$ , whose morphisms are the degree zero graded  $A^\natural$ -linear homomorphisms of such modules. As  $\mathbf{Gr}(A^\natural)$  is abelian, we can speak of its category  $\mathbf{ChGr}(A^\natural)$  of complexes. In particular, the homologically bounded complexes in this category, along with their morphisms, form a full subcategory of  $\mathbf{ChGr}(A^\natural)$ , which we denote by  $\mathbf{Ch}^b \mathbf{Gr}(A^\natural)$ .

**Lemma 1.4.1.** *Let  $A$  be a graded algebra. For every complex  $X$  in  $\mathbf{ChGr}(A)$ , there exists a complex  $F$  of graded free  $A$ -modules such that  $X \simeq F$  in  $\mathbf{ChGr}(A)$ . Moreover, if  $X$  is in  $\mathbf{Ch}^b \mathbf{Gr}(A)$ , then  $F_i = 0$  for all  $i \ll 0$ .*

As a result of this lemma, the derived category  $\mathbf{DGr}(A)$  can be constructed as the category whose objects are complexes of graded free  $A^\natural$ -modules, and whose morphisms are the homotopy classes of morphisms of these complexes.

*Remark 1.4.2.* Given a DG algebra  $A = A^\natural$  with trivial differential, there is no harm in denoting the category of graded  $A$ -modules by  $\mathbf{Gr}(A)$ . We shall also frequently use this convention for the respective complex and derived categories.

It is clear that over an arbitrary DG algebra  $A$ , the derived categories  $\mathbf{DDG}(A)$  and  $\mathbf{DGr}(A^\natural)$  are quite different. Indeed, the objects of  $\mathbf{DDG}(A)$  have a richer structure than those of  $\mathbf{DGr}(A^\natural)$ . To see this, notice that the objects of  $\mathbf{DG}(A)$  incorporate two differentials: one belonging to  $A$ , and another which is inherent of the particular DG  $A$ -module. However, the situation is not so transparent in the case that  $A$  is simply a graded algebra. It is in this setting that we are interested in determining when the category  $\mathbf{DDG}(A)$  can be naturally obtained from the category  $\mathbf{DGr}(A)$ .

**1.5. Totaling.** We now introduce a natural functor with which we will compare the derived categories of interest. Throughout this section, we suppose that  $A$  is a graded algebra.<sup>1</sup>

Let  $X$  be a cochain complex in  $\mathbf{ChGr}(A)$ , and denote the internal grading of each  $X^i$  with a subscript; that is,  $X^i = \bigoplus_{j \in \mathbb{Z}} X_j^i$ . We define the *totaling* of  $X$  to be the complex  $\text{Tot } X = ((\text{Tot } X)^\natural, \partial^{\text{Tot } X})$ , whose underlying graded structure is given by

$$(\text{Tot } X)^\natural = \bigoplus_{i \in \mathbb{Z}} \Sigma^{-i} X^i.$$

Therefore, the  $d^{\text{th}}$  homological component of  $\text{Tot } X$  is given by

$$(\text{Tot } X)_d = \bigoplus_{i \in \mathbb{Z}} X_{d+i}^i$$

and we define the  $A$ -module structure on  $\text{Tot } X$  by

$$a(\sigma^{-i} x^i)_{i \in \mathbb{Z}} = \left( \sigma^{-i}((-1)^{|a||i|} a x^i) \right)_{i \in \mathbb{Z}}$$

for each  $a \in A$  and  $(\sigma^{-i} x^i)_{i \in \mathbb{Z}} \in \text{Tot } X$ . Moreover, the differential on  $\text{Tot } X$  is defined to be

$$\partial^{\text{Tot } X}((\sigma^{-i} x^i)_{i \in \mathbb{Z}}) = (\sigma^{-i-1} \partial_X^i(x^i))_{i \in \mathbb{Z}}$$

for each  $(\sigma^{-i} x^i)_{i \in \mathbb{Z}} \in \text{Tot } X$ .

It is now easy to check that  $\text{Tot } X$  is in fact a DG module over  $A$ . Indeed, for any  $a \in A$  and  $(\sigma^{-i} x^i)_{i \in \mathbb{Z}} \in \text{Tot } X$ , we have

$$\begin{aligned} \partial^{\text{Tot } X}(a(\sigma^{-i} x^i)_{i \in \mathbb{Z}}) &= \partial^{\text{Tot } X} \left( \left( \sigma^{-i}((-1)^{|a||i|} a x^i) \right)_{i \in \mathbb{Z}} \right) \\ &= \left( \sigma^{-i-1} \partial_X^i((-1)^{|a||i|} a x^i) \right)_{i \in \mathbb{Z}} \\ &= \left( (-1)^{|a||i|} \sigma^{-i-1}(a \partial_X^i(x^i)) \right)_{i \in \mathbb{Z}} \\ &= \left( (-1)^{|a||i|} (-1)^{|a|(|i|+1)} a \sigma^{-i-1} \partial_X^i(x^i) \right)_{i \in \mathbb{Z}} \\ &= (-1)^{|a||i|} a (\sigma^{-i-1} \partial_X^i(x^i))_{i \in \mathbb{Z}} \\ &= (-1)^{|a||i|} a \partial^{\text{Tot } X}((\sigma^{-i} x^i)_{i \in \mathbb{Z}}) \end{aligned}$$

so that the Leibniz rule is satisfied.

If  $\mu : X \rightarrow Y$  is a morphism of complexes of graded  $A$ -modules, we define a map:

$$\begin{aligned} \text{Tot } \mu : \text{Tot } X &\rightarrow \text{Tot } Y \\ (\sigma^{-i} x^i)_{i \in \mathbb{Z}} &\mapsto (\sigma^{-i} \mu^i(x^i))_{i \in \mathbb{Z}} \end{aligned}$$

To see that  $\text{Tot } \mu$  is a morphism of DG  $A$ -modules, note that for each  $x^i \in X_j^i$ ,  $\mu^i(x^i) \in Y_j^i$ . Therefore, totaling defines a functor  $\text{Tot} : \mathbf{ChGr}(A) \rightarrow \mathbf{DG}(A)$ .

*Remark 1.5.1.* In order to define the totaling of a *chain* complex in  $\mathbf{ChGr}(A)$ , one should note that  $\bigoplus_{i \in \mathbb{Z}} \Sigma^{-i} X^i \cong \bigoplus_{i \in \mathbb{Z}} \Sigma^i X_i$ .

---

<sup>1</sup>Indeed,  $\text{Tot}$  can be defined on the category  $\mathbf{ChDG}(A)$  over an arbitrary DG algebra  $A$  (see [1, Section 7.2] for details). However, for the purposes of our work, it suffices to define the functor in the present setting.



We now state and prove several results that demonstrate some nice properties of the totaling functor.

**Fact 1.5.2.** *Let  $A$  be a graded algebra. If  $X \in \mathbf{ChGr}(A)$  is such that each  $X_i$  is free as a graded left  $A$ -module and  $X_i = 0$  for all  $i \ll 0$ , then  $\text{Tot } X$  is semifree in  $\text{DG}(A)$ .*

*Proof.* Let  $\ell \in \mathbb{Z}$  be such that  $X_i = 0$  for all  $i < \ell$ . Now, for every  $j \geq \ell$ , let  $E_j$  be a well-ordered basis for  $X_j$  over  $A$ , and define

$$E = \bigsqcup_{j=\ell}^{\infty} \Sigma^j E_j.$$

For  $e_j \in E_j$  and  $e_{j'} \in E_{j'}$ , let  $\sigma^j e_j < \sigma^{j'} e_{j'}$  if either  $j < j'$ , or  $j = j'$  and  $e_j < e_{j'}$  in  $E_j = E_{j'}$ . Then, clearly,  $E$  defines a well-ordered basis for  $(\text{Tot } X)^{\natural}$  over  $A$ . Furthermore, for each  $e_j \in E_j$  we have

$$\partial^{\text{Tot } X}(\sigma^j e_j) = \sigma^{j-1} \partial_j^X(e_j) \subseteq \sigma^{j-1} A(\{e_{j-1} \in E_{j-1}\})$$

which certainly implies that  $E$  is a semibasis of  $\text{Tot } X$  over  $A$ .  $\square$

Below we will illustrate the behavior of  $\text{Tot}$  with respect to tensor products. First, however, we discuss general facts about tensor products of the respective objects.

**Facts 1.5.3.** Let  $A$  be an arbitrary DG algebra, and suppose that  $X$  and  $Y$  are (cochain) complexes of  $A^{\natural}$ -modules such that  $X$  admits an  $A^{\natural}$ -bimodule structure. Recall that the tensor product  $X \otimes_{A^{\natural}} Y$  is also a complex of  $A^{\natural}$ -modules, defined by

$$(X \otimes_{A^{\natural}} Y)^i = \bigoplus_{j+\ell=i} (X^j \otimes_{A^{\natural}} Y^{\ell})$$

$$\partial_{X \otimes Y}^i(x \otimes y) = \partial_X^j(x) \otimes y + (-1)^j x \otimes \partial_Y^{\ell}(y)$$

for each  $x \in X^j$  and  $y \in Y^{\ell}$ . Moreover, the following hold.

- (1) If  $X^j$  and  $Y^{\ell}$  are graded  $A^{\natural}$ -modules, for each  $j, \ell \in \mathbb{Z}$ , then  $X \otimes_{A^{\natural}} Y$  is a complex of graded  $A^{\natural}$ -modules. Note that each homological degree of this complex is actually a graded tensor product. That is, for each  $j, \ell \in \mathbb{Z}$ ,  $X^j \otimes_{A^{\natural}} Y^{\ell}$  is a tensor product on the internal grading of  $X^j$  and  $Y^{\ell}$ . To avoid confusion, we shall denote this tensor product by  $\otimes_{\text{Gr}(A^{\natural})}$ . Furthermore, the tensor product at the complex level shall be represented with the symbol  $\otimes_{\mathbf{ChGr}(A^{\natural})}$ .
- (2) If  $M$  and  $N$  are DG  $A$ -modules, then the action of  $A$  on the underlying complex  $M \otimes_{A^{\natural}} N$  defined by

$$a(m \otimes n) = (am) \otimes n$$

for each  $a \in A$  and  $m \otimes n \in M \otimes_{A^{\natural}} N$ , endows  $M \otimes_{A^{\natural}} N$  with the structure of a DG  $A$ -module. So as not to allow confusion, when we wish for the tensor product of DG  $A$ -modules to possess the same amount of structure, we shall use the symbol  $\otimes_{\text{DG}(A)}$  to denote it.

**Proposition 1.5.4.** *Let  $A$  be a graded algebra, and suppose that  $X, Y \in \mathbf{ChGr}(A)$ , where  $X$  has an  $A$ -bimodule structure. Then there exists a natural isomorphism*

$$\mathrm{Tot}(X \otimes_{\mathbf{ChGr}(A)} Y) \cong \mathrm{Tot} X \otimes_{\mathrm{DG}(A)} \mathrm{Tot} Y$$

of DG  $A$ -modules.

*Proof.* Consider the degree zero chain map of DG  $A$ -modules defined by the family

$$\begin{aligned} \lambda_d : \mathrm{Tot}(X \otimes_{\mathbf{ChGr}(A)} Y)_d &\rightarrow (\mathrm{Tot} X \otimes_{\mathrm{DG}(A)} \mathrm{Tot} Y)_d \\ \sigma^{-j-\ell}(x \otimes y) &\mapsto (-1)^{\ell|x|} \sigma^{-j} x \otimes \sigma^{-\ell} y \end{aligned}$$

for each  $x \in X^j$ ,  $y \in Y^\ell$ , and all  $j, \ell \in \mathbb{Z}$ . Clearly, each  $\lambda_d$  is an isomorphism of graded  $A$ -modules. To see that  $\lambda = (\lambda_d)$  is furthermore a morphism, we show that the following square commutes for each  $d \in \mathbb{Z}$ .

$$\begin{array}{ccc} \mathrm{Tot}(X \otimes_{\mathbf{ChGr}(A)} Y)_d & \xrightarrow{\partial_d^{\mathrm{Tot}(X \otimes Y)}} & \mathrm{Tot}(X \otimes_{\mathbf{ChGr}(A)} Y)_{d-1} \\ \lambda_d \downarrow & & \downarrow \lambda_{d-1} \\ (\mathrm{Tot} X \otimes_{\mathrm{DG}(A)} \mathrm{Tot} Y)_d & \xrightarrow{\partial_d^{\mathrm{Tot} X \otimes \mathrm{Tot} Y}} & (\mathrm{Tot} X \otimes_{\mathrm{DG}(A)} \mathrm{Tot} Y)_{d-1} \end{array}$$

Fixing  $j, \ell \in \mathbb{Z}$ , we have the following for all  $x \in X^j$  and  $y \in Y^\ell$ :

$$\begin{aligned} &\lambda_{d-1} \circ \partial_d^{\mathrm{Tot}(X \otimes Y)}(\sigma^{-j-\ell}(x \otimes y)) \\ &= \lambda_{d-1} \left( \sigma^{-j-\ell-1} \partial_{X \otimes Y}^{j+\ell}(x \otimes y) \right) \\ &= \lambda_{d-1} \left( \sigma^{-j-\ell-1} \left( \partial_X^j(x) \otimes y + (-1)^j x \otimes \partial_Y^\ell(y) \right) \right) \\ &= \lambda_{d-1} \left( \sigma^{-j-\ell-1} \left( \partial_X^j(x) \otimes y \right) + (-1)^j \sigma^{-j-\ell-1} (x \otimes \partial_Y^\ell(y)) \right) \\ &= (-1)^{\ell|x|} \sigma^{-j-1} \partial_X^j(x) \otimes \sigma^{-\ell} y + (-1)^{j+(\ell+1)|x|} \sigma^{-j} x \otimes \sigma^{-\ell-1} \partial_Y^\ell(y) \end{aligned}$$

Furthermore, for the same choice of  $x$  and  $y$ , the alternate composition yields:

$$\begin{aligned} &\partial_d^{\mathrm{Tot} X \otimes \mathrm{Tot} Y} \circ \lambda_d(\sigma^{-j-\ell}(x \otimes y)) \\ &= \partial_d^{\mathrm{Tot} X \otimes \mathrm{Tot} Y}((-1)^{\ell|x|} \sigma^{-j} x \otimes \sigma^{-\ell} y) \\ &= (-1)^{\ell|x|} \left( \partial_{|x|+j}^{\mathrm{Tot} X}(\sigma^{-j} x) \otimes \sigma^{-\ell} y + (-1)^{|x|+j} \sigma^{-j} x \otimes \partial_{|y|+\ell}^{\mathrm{Tot} Y}(\sigma^{-\ell} y) \right) \\ &= (-1)^{\ell|x|} \left( \sigma^{-j-1} \partial_X^j(x) \otimes \sigma^{-\ell} y + (-1)^{|x|+j} \sigma^{-j} x \otimes \sigma^{-\ell-1} \partial_Y^\ell(y) \right) \\ &= (-1)^{\ell|x|} \sigma^{-j-1} \partial_X^j(x) \otimes \sigma^{-\ell} y + (-1)^{j+(\ell+1)|x|} \sigma^{-j} x \otimes \sigma^{-\ell-1} \partial_Y^\ell(y) \end{aligned}$$

The result is immediate.  $\square$

**Proposition 1.5.5.** *Let  $A$  be a graded algebra, and furthermore suppose that  $X \in \mathbf{ChGr}(A)$  and  $Y \in \mathbf{Ch}^b \mathbf{Gr}(A)$ . If  $X$  admits an  $A$ -bimodule structure, then for each  $j \in \mathbb{Z}$ , there exists an isomorphism*

$$\mathrm{Tor}_j^{\mathrm{DG}(A)}(\mathrm{Tot} X, \mathrm{Tot} Y) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Tor}_i^{\mathbf{ChGr}(A)}(X, Y)_{j-i}.$$

*Proof.* By Lemma 1.4.1, there exists a chain complex  $F$  of graded free  $A$ -modules, with  $F_i = 0$  for all  $i \ll 0$ , such that  $F \simeq Y$ . Then Lemma 1.5.2 implies that  $\mathrm{Tot} F$

is semifree, and thus a projective object in  $\mathbf{DG}(A)$ . This, along with Proposition 1.5.4, yield the following isomorphisms:

$$\begin{aligned} \mathrm{Tor}_j^{\mathbf{DG}(A)}(\mathrm{Tot} X, \mathrm{Tot} Y) &\cong H_j(\mathrm{Tot} X \otimes_{\mathbf{DG}(A)} \mathrm{Tot} F) \\ &\cong H_j(\mathrm{Tot}(X \otimes_{\mathbf{ChGr}(A)} F)) \\ &\cong \bigoplus_{i \in \mathbb{Z}} H_i(X \otimes_{\mathbf{ChGr}(A)} F)_{j-i} \\ &\cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Tot}_i^{\mathbf{ChGr}(A)}(X, Y)_{j-i} \end{aligned}$$

□

**Fact 1.5.6.** *The functor  $\mathrm{Tot}$  preserves quasiisomorphism, and therefore extends to a functor of derived categories*

$$\mathrm{Tot} : \mathbf{DGr}(A) \rightarrow \mathbf{DDG}(A)$$

for any graded algebra  $A$ .

*Proof.* Let  $\mu : X \rightarrow Y$  be a quasiisomorphism of cochain complexes in  $\mathbf{ChGr}(A)$ . Then passing the induced morphism  $\mathrm{Tot} \mu : \mathrm{Tot} X \rightarrow \mathrm{Tot} Y$  of  $\mathbf{DG} A$ -modules to homology yields the map

$$H(\mathrm{Tot} \mu) : \bigoplus_{i \in \mathbb{Z}} \Sigma^{-i} H^i(X) \rightarrow \bigoplus_{i \in \mathbb{Z}} \Sigma^{-i} H^i(Y)$$

which is defined by  $(\sigma^{-i} z^i)_{i \in \mathbb{Z}} \rightarrow (\sigma^{-i} \mu(z^i))_{i \in \mathbb{Z}}$  for each cycle  $z^i \in H^i(X)$ . Since  $H\mu$  is an isomorphism of graded  $A$ -modules, so is  $H(\mathrm{Tot} \mu)$ . The result follows. □

## 2. THE IMAGE OF TOT

In this section, we turn our attention to the image of the totaling functor on the derived categories. Until otherwise stated, we shall assume that  $A$  is a graded algebra. Furthermore, all polynomial rings shall be standard graded.

**Lemma 2.0.1.** *Let  $A$  be an arbitrary  $DG$  algebra, and  $X \in \mathbf{ChGr}(A^\natural)$ . If  $H(X) = H_n(X)$  for some  $n \in \mathbb{Z}$ , then  $X \simeq H(X)$ .*

*Proof.* Consider the following commutative diagram of complexes and chain maps between them

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X_{d+1} & \xrightarrow{\partial_{d+1}} & X_d & \xrightarrow{\partial_d} & X_{d-1} \longrightarrow \cdots \\ & & \parallel & & \uparrow \iota & & \uparrow \\ \cdots & \longrightarrow & X_{d+1} & \xrightarrow{\partial_{d+1}} & Z_d(X) & \longrightarrow & 0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow \pi & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & H_d(X) & \longrightarrow & 0 \longrightarrow \cdots \end{array}$$

where  $\iota$  and  $\pi$  are the obvious inclusion and projection maps, respectively. Clearly, these chain maps induce an isomorphism in homology; thus  $X \simeq H(X)$ . □

The following lemma is a result of Avramov and Jorgensen. For the reader's convenience, we include the proof.

**Lemma 2.0.2.** [2] *Let  $A$  be an augmented non-negatively graded  $k$ -algebra, and suppose that  $M$  is a semifree DG  $A$ -module with  $\partial(M_i) \subseteq AM_{i-1}$  such that  $M \simeq \text{Tot } X$  for some chain complex  $X$  of graded  $A$ -modules. If  $H(M)$  is indecomposable over  $A$ , then*

$$\text{rank}_A M = \sum_{i \in \mathbb{N}} \beta_i^{H(M)}$$

where  $\beta_i^{H(M)}$  is the  $i^{\text{th}}$  total Betti number of  $H(M)$ ; that is,  $\beta_i^{H(M)} = \sum_{j \in \mathbb{Z}} \beta_{i,j}$  for a minimal graded free resolution

$$\cdots \rightarrow \bigoplus_{j \in \mathbb{Z}} \Sigma^j A^{\beta_{2,j}} \rightarrow \bigoplus_{j \in \mathbb{Z}} \Sigma^j A^{\beta_{1,j}} \rightarrow \bigoplus_{j \in \mathbb{Z}} \Sigma^j A^{\beta_{0,j}} \rightarrow H(M) \rightarrow 0$$

of  $H(M)$  over  $A$ .

*Proof.* The fact that  $M \simeq \text{Tot } X$  produces the following isomorphisms of graded  $A$ -modules.

$$H(M) \cong H(\text{Tot } X) \cong \bigoplus_{i \in \mathbb{Z}} \Sigma^i H_i(X)$$

Now since we have assumed that  $H(M)$  does not decompose over  $A$ , it follows that  $H(X) \cong H_d(X)$  for some  $d \in \mathbb{Z}$ , and therefore  $X \simeq H(X)$  by Lemma 2.0.1. Furthermore recalling that totalizing preserves quasiisomorphism, the following holds.

$$(2.0.2.1) \quad M \simeq \text{Tot } X \simeq \text{Tot } H(X)$$

Since the homology of  $X$  is concentrated in degree  $d$ , we obtain

$$(2.0.2.2) \quad \text{Tot } H(X) \cong \text{Tot } H_d(X) = \Sigma^n H_d(X) \cong H(M).$$

However, noting that the terms in (2.0.2.2) are merely graded  $A$ -modules, we have that  $\text{Tot } H(X) \simeq H(M)$ . Combining this fact with (2.0.2.1) yields  $M \simeq H(M)$ . This quasiisomorphism and Proposition 1.5.5 account for the isomorphisms below; the first equality follows from the minimality of  $M$ , and the second is a consequence of the semifreeness of  $M$ .

$$\begin{aligned} M \otimes_{\text{DG}(A)} k &= H(M \otimes_{\text{DG}(A)} k) \\ &= \bigoplus_{j \in \mathbb{Z}} \text{Tor}_j^{\text{DG}(A)}(M, k) \\ &\cong \bigoplus_{j \in \mathbb{Z}} \text{Tor}_j^{\text{DG}(A)}(H(M), k) \\ &\cong \bigoplus_{i,j \in \mathbb{Z}} \text{Tor}_i^{\text{ChGr}(A)}(H(M), k)_{j-i} \end{aligned}$$

Hence, we arrive at

$$\begin{aligned} \text{rank}_A M &= \text{rank}_k(M \otimes_{\text{DG}(A)} k) \\ &= \sum_{i,j \in \mathbb{Z}} \text{rank}_k \text{Tor}_j^{\text{DG}(A)}(H(M), k)_{j-i} \\ &= \sum_{i \in \mathbb{Z}} \beta_i^{H(M)} \end{aligned}$$

which is what was to be proved.  $\square$

**2.1. Over**  $A = k[x_1, \dots, x_d]$ . We shall now investigate the image of Tot on the derived categories in the case that  $A$  is a polynomial ring in more than one variable.

**Theorem 2.1.1.** *Let  $A = k[x_1, \dots, x_d]$  where  $d \geq 2$ . Then the functor*

$$\text{Tot} : \mathbf{DGr}(A) \rightarrow \mathbf{DDG}(A)$$

*is not onto.*

*Proof.* Consider the rank four semifree DG module  $M$  over  $A = k[x_1, \dots, x_d]$ , where  $d \geq 2$ , given by  $M^\natural = Ae_1 \oplus Ae_2 \oplus Ae_3 \oplus Ae_4$  such that  $|e_1| = 0$ ,  $|e_2| = 3$ ,  $|e_3| = 4$ ,  $|e_4| = 8$ , and

$$\begin{aligned} \partial(e_1) &= 0 \\ \partial(e_2) &= x_1 x_2 e_1 \\ \partial(e_3) &= x_2^3 e_1 \\ \partial(e_4) &= x_1^7 e_1 - x_2^4 e_2 + x_1 x_2^2 e_3. \end{aligned}$$

We will show that there does not exist a complex of graded  $A$ -modules whose totaling is quasiisomorphic to  $M$ .

First, note that minimal generating sets of  $Z(M)$  and  $B(M)$  over  $A$  are  $\{e_1, x_2^2 e_2 - x_1 e_3\}$  and  $\{x_1 x_2 e_1, x_2^3 e_1, x_1^7 e_1 - x_2^4 e_2 + x_1 x_2^2 e_3\}$ , respectively. Thus, a presentation matrix of  $H(M)$  is given by the following.

$$\begin{bmatrix} x_1 x_2 & x_2^3 & x_1^7 \\ 0 & 0 & -x_2^2 \end{bmatrix}$$

One can now check that  $H(M)$  is indecomposable over  $A$ . Writing down the deleted minimal graded free resolution of  $H(M)$  over  $A$  yields

$$0 \rightarrow \Sigma^4 A \xrightarrow{\begin{bmatrix} x_2^2 \\ -x_1 \\ 0 \end{bmatrix}} \Sigma^2 A \oplus \Sigma^3 A \oplus \Sigma^7 A \xrightarrow{\begin{bmatrix} x_1 x_2 & x_2^3 & x_1^7 \\ 0 & 0 & -x_2^2 \end{bmatrix}} A \oplus \Sigma^5 A \rightarrow 0$$

whence we arrive at

$$\sum_{i \in \mathbb{Z}} \beta_i^{H(M)} = 2 + 3 + 1 = 6 \neq 4 = \text{rank}_A(M).$$

The result follows by Lemma 2.0.2.  $\square$

It is now natural to wonder what characteristic of the DG module in the previous proof causes it to fall outside of the image of Tot. A partial answer to this question is given in the following section, where we shall furthermore define a special class of semifree DG modules over an arbitrary DG algebra with trivial differential which are always contained in the image of Tot.

**2.2. Crossing.** In this section, we discuss a particular class of DG  $A$ -modules which are characterized by the structure of their cycles.

**Definition 2.2.1.** Let  $A$  be an arbitrary DG algebra, and  $M$  a semifree DG module with semibasis  $E$  over  $A$ . Define a family of disjoint sets recursively by

$$\begin{aligned} \mathcal{E}_0 &= \{e \in E \mid \partial(e) = 0\} \\ (2.2.1.1) \quad \mathcal{E}_\ell &= \{e \in E \mid 0 \neq \partial(e) \in A\mathcal{E}_{\ell-1}\} \end{aligned}$$

for all  $\ell \in \mathbb{Z}^+$ . Notice that since  $M$  is semifree,  $\mathcal{E}_0$  is nonempty. Now consider the following containment of sets.

$$(2.2.1.2) \quad \bigsqcup_{\ell \in \mathbb{N}} \mathcal{E}_\ell \subseteq E$$

If the containment in (2.2.1.2) is strict, we say that  $E$  has *crossing*; otherwise, we say that  $E$  has *no crossing*.

To illustrate this concept, we provide two examples.

**Example 2.2.2.** Let  $A = k[x, y, z]$  and consider the rank four semifree DG  $A$ -module  $M$  with semibasis  $E = \{e_1, e_2, e_3, e_4\}$  such that  $|e_1| = 0$ ,  $|e_2| = 2$ ,  $|e_3| = 3$ ,  $|e_4| = 5$ , and

$$\begin{aligned} \partial(e_1) &= 0 \\ \partial(e_2) &= xe_1 \\ \partial(e_3) &= yze_1 \\ \partial(e_4) &= xz^3e_1 + yze_2 - xe_3. \end{aligned}$$

Then  $\mathcal{E}_0 = \{e_1\}$ ,  $\mathcal{E}_1 = \{e_2, e_3\}$ , and  $\mathcal{E}_\ell = \emptyset$  for  $\ell \geq 2$ . Thus, the strict containment

$$\bigsqcup_{\ell \in \mathbb{N}} \mathcal{E}_\ell \subsetneq E$$

implies that  $E$  has crossing.

**Example 2.2.3.** Let  $A$  and  $M$  be as in the previous example; however, express  $M$  in terms of the semibasis  $E'$  defined by  $e'_i = e_i$  for  $1 \leq i \leq 3$  and  $e'_4 = e_4 - z^3e_2$ . The action of  $\partial^M$  on  $E'$  is given by

$$\begin{aligned} \partial(e'_1) &= 0 \\ \partial(e'_2) &= xe'_1 \\ \partial(e'_3) &= yze'_1 \\ \partial(e'_4) &= yze'_2 - xe'_3. \end{aligned}$$

In this case, we obtain  $\mathcal{E}_0 = \{e'_1\}$ ,  $\mathcal{E}_1 = \{e'_2, e'_3\}$ ,  $\mathcal{E}_2 = \{e'_4\}$ , and  $\mathcal{E}_\ell = \emptyset$  for  $\ell \geq 3$ . Therefore, we have an equality of sets

$$\bigsqcup_{\ell \in \mathbb{N}} \mathcal{E}_\ell = E'$$

which implies that  $E$  has no crossing.

*Remark 2.2.4.* The DG module exhibited in the previous two examples might lead one to believe that any DG module can be expressed in terms of a semibasis which has no crossing; however, this is not the case. Consider, for example, the DG module illustrated in the proof of Theorem 2.1.1. The following result shows that it is impossible to express this DG module in terms of a semibasis which is devoid of crossing.

**Theorem 2.2.5.** *Let  $M$  be a semifree DG module over a graded algebra  $A$ . Then  $M$  possesses a semibasis without crossing if and only if there exists a complex  $X$  of graded free  $A$ -modules such that  $\text{Tot } X = M$ .*

*Proof.* Let  $E = \bigsqcup_{d \in \mathbb{N}} E_d$  be a semibasis of  $M$  over  $A$ . First we suppose that  $E$  has no crossing, implying that  $E = \bigsqcup_{\ell \in \mathbb{N}} \mathcal{E}_\ell$ , where

$$\begin{aligned}\mathcal{E}_0 &= \{e \in E \mid \partial^M(e) = 0\} \\ \mathcal{E}_\ell &= \{e \in E \mid 0 \neq \partial^M(e) \in A\mathcal{E}_{\ell-1}\}\end{aligned}$$

for all  $\ell \in \mathbb{Z}^+$ . Now define a (possibly infinite) sequence  $X$  of homomorphisms of graded  $A$ -modules by

$$X : \quad \cdots \rightarrow \Sigma^{-2}A\mathcal{E}_2 \xrightarrow{\partial_2^X} \Sigma^{-1}A\mathcal{E}_1 \xrightarrow{\partial_1^X} A\mathcal{E}_0 \rightarrow 0$$

where, for each  $\ell \in \mathbb{N}$  and  $e \in \mathcal{E}_\ell$ , we have that

$$\partial_\ell^X((0, \dots, 0, \sigma^{-\ell}e, 0, \dots, 0)) = \sigma^{-\ell+1}\partial^M(e) \subseteq \Sigma^{-\ell+1}A\mathcal{E}_{\ell-1}.$$

By construction,  $\partial_{\ell+1}^X \circ \partial_\ell^X = 0$  for all  $\ell \in \mathbb{N}$ , implying that  $X \in \mathbf{ChGr}(A)$ . To see that  $\text{Tot } X = M$ , notice that

$$\begin{aligned}(\text{Tot } X)^\natural &= \bigoplus_{\ell \in \mathbb{N}} \Sigma^\ell X_\ell \\ &= \bigoplus_{\ell \in \mathbb{N}} \Sigma^\ell \Sigma^{-\ell} A\mathcal{E}_\ell \\ &\cong \bigoplus_{\ell \in \mathbb{N}} A\mathcal{E}_\ell \\ &= M^\natural\end{aligned}$$

and that

$$\begin{aligned}\partial^{\text{Tot } X}(\sigma^\ell \sigma^{-\ell}e) &= \sigma^{\ell-1} \partial_\ell^X(\sigma^{-\ell}e) \\ &= \sigma^{\ell-1}(\sigma^{-\ell+1} \partial^M(e)) \\ &= \partial^M(e)\end{aligned}$$

for all  $\ell \in \mathbb{N}$  and  $e \in \mathcal{E}_\ell$ . The result follows.

On the other hand, suppose that  $X$ , expressed below, is a chain complex of graded free  $A$ -modules such that  $\text{Tot } X = M$ .

$$X : \quad \cdots \rightarrow X_{i+1} \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \rightarrow \cdots$$

Then we can write

$$A \left( \bigsqcup_{d \in \mathbb{N}} E_d \right) = AE = M^\natural = (\text{Tot } X)^\natural = \bigoplus_{i \in \mathbb{Z}} \Sigma^i X_i.$$

Notice that since  $M$  is semifree, it must be true that  $\partial^M(AE_0) = 0$ . This implies that the quantity given by

$$n = \max\{\ell \in \mathbb{Z} \mid \partial_i^X \text{ acts trivially for all } i \leq \ell\}$$

is certainly well-defined whenever  $\partial^X \neq 0$ .

Now let  $\bigsqcup_{\ell \in \mathbb{N}} \mathcal{E}_\ell$  be a partitioning of  $E$  such that the following holds.

$$A\mathcal{E}_\ell = \begin{cases} \bigoplus_{i \in \mathbb{N}} \Sigma^{n-i} X_{n-i} & \text{for } \ell = 0 \\ \Sigma^{n+\ell} X_{n+\ell} & \text{for } \ell > 0 \end{cases}$$

It suffices to show that  $\bigsqcup_{\ell \in \mathbb{N}} \mathcal{E}_\ell$  satisfies the conditions of (2.2.1.1). It is obvious that  $\mathcal{E}_0$  fits the bill. Thus, let  $\ell \in \mathbb{Z}^+$ , and notice that for each  $e \in \mathcal{E}_\ell$ ,  $\partial^M(e) = \sigma^{\ell-1} \partial^X(e)$ . This yields

$$\mathcal{E}_\ell = \{e \in E \mid \partial^M(e) = \sigma^{\ell-1} \partial^X(e) \in \Sigma^{\ell-1} X_{\ell-1} = A\mathcal{E}_{\ell-1}\}$$

and the result follows.  $\square$

This result demonstrates precisely when a semifree module is in the image of  $\text{Tot} : \mathbf{ChGr}(A) \rightarrow \mathbf{DG}(A)$  whenever  $A$  is a graded algebra. We shall illustrate with an example.

**Example 2.2.6.** Let  $M$  be the rank four semifree DG module with semibasis  $E$  over  $A = k[x, y, z]$  defined in Example 2.2.3. As  $E$  has no crossing, the construction in the previous proof yields a complex

$$X : \quad 0 \rightarrow \Sigma^{-2} A e_4 \xrightarrow{\begin{bmatrix} yz \\ -x \end{bmatrix}} \Sigma^{-1} (A e_2 \oplus A e_3) \xrightarrow{\begin{bmatrix} x & yz \end{bmatrix}} A e_1 \rightarrow 0$$

of graded  $A$ -modules such that  $\text{Tot } X = M$ .

**Corollary 2.2.7.** *Let  $A$  be a graded domain, and suppose that  $M$  is a semifree DG  $A$ -module. If  $\text{rank}_A M \leq 3$  then there exists a complex  $X$  of graded  $A$ -modules such that  $\text{Tot } X = M$ .*

*Proof.* Let  $E = \{e_1, \dots, e_d\}$  be a well-ordered for  $M^\natural$  over  $A$ . Since  $M$  is semifree, we obtain the following possible (non-trivial) forms for its differential, where  $a_{ij} \in A$  for each  $i, j$ .

$$\begin{aligned} d = 1 & \Rightarrow \partial(e_1) = 0 \\ d = 2 & \Rightarrow \begin{aligned} \partial(e_1) &= 0 \\ \partial(e_2) &= a_{12}e_1 \end{aligned} \\ d = 3 & \Rightarrow \begin{aligned} \partial(e_1) &= 0 & \partial(e_1) &= 0 \\ \partial(e_2) &= a_{12}e_1 & \text{or} & \partial(e_2) = 0 \\ \partial(e_3) &= a_{13}e_1 & \partial(e_3) &= a_{13}e_1 + a_{23}e_2 \end{aligned} \end{aligned}$$

Note that in each case,  $E$  has no crossing. The result follows by Theorem 2.2.5.  $\square$

**2.3. Over  $A = k[x]$ .** We now turn our attention to the image of  $\text{Tot} : \mathbf{DGr}(A) \rightarrow \mathbf{DDG}(A)$  in the case that  $A$  is a polynomial ring in one variable.

**Lemma 2.3.1.** *Let  $M$  be a DG module which is  $n$ -generated over  $A = k[x]$ . Then there exist integers  $0 \leq s \leq d$  and  $1 \leq t \leq d$  such that  $H(M)$  has a graded minimal free resolution over  $A$  given by*

$$0 \rightarrow \bigoplus_{j=1}^s \Sigma^{c_j} A \xrightarrow{\begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_s \\ \hline & & 0 \end{bmatrix}} \bigoplus_{i=1}^t \Sigma^{r_i} A \rightarrow H(M) \rightarrow 0$$



for some integers  $r_i, c_j$  for  $1 \leq i \leq t$  and  $1 \leq j \leq s$ , and where each  $h_i = x^{c_i - r_i} \in Ax$  for each  $1 \leq i \leq s$ .

*Proof.* Notice that we can assume that  $H(M)$  is not a free  $A$ -module, since otherwise  $s = 0$  and the result is obvious. Now, since  $B(M) \subseteq Z(M)$  are submodules of a finitely generated module over a principal ideal domain, it follows that  $Z(M)$  and  $B(M)$  are finitely generated  $A$ -modules; the same must also be true of  $H(M)$ . By Hilbert's syzygy theorem, any minimal free resolution of  $H(M)$  has at most length one. Therefore, consider the following minimal free resolution of  $H(M)$  over  $A$

$$F : \quad 0 \rightarrow \bigoplus_{j=1}^s \Sigma^{\tilde{c}_j} A \xrightarrow{\varphi} \bigoplus_{i=1}^t \Sigma^{\tilde{r}_i} A \rightarrow H(M) \rightarrow 0$$

where  $\tilde{r}_i$  is the degree of the  $i^{\text{th}}$  minimal generator of  $H(M)$ . The exactness of  $F$  tells us that  $s \leq t$ . Furthermore, the minimality of  $F$  and homogeneity of  $\varphi$  yield that each nonzero entry of  $\varphi = (f_{i,j})$  is such that  $\tilde{r}_i - \tilde{c}_j = |f_{i,j}| \geq 1$ . Manipulation of the indices in this relation reveals that

$$\begin{aligned} |f_{i+1,j}| - |f_{i,j}| &= |f_{i+1,j+1}| - |f_{i,j+1}| \\ |f_{i,j+1}| - |f_{i,j}| &= |f_{i+1,j+1}| - |f_{i+1,j}| \end{aligned}$$

for each  $i, j$ . Therefore, it follows that we can rearrange the rows and columns of  $\varphi$  so that its nonzero entries satisfy

$$(2.3.1.1) \quad |f_{i,j}| \leq |f_{i',j}| \quad \text{and} \quad |f_{i,j}| \leq |f_{i,j'}|$$

for all  $i \leq i'$  and  $j \leq j'$ . That is, it suffices to assume that any nonzero entry of  $\varphi$  has only zeros and entries of higher degree in each spot below and to the right of it.

The proof will continue recursively for  $1 \leq \ell \leq s$ . Notice that by the minimality of  $F$ , there exists a nonzero entry in column  $\ell$ ; let  $i'$  be the smallest integer such that  $f_{i',\ell} \neq 0$ . Shift the rows of  $\varphi$  so that  $f_{i',\ell} \mapsto f_{\ell,\ell}$  and  $f_{i,\ell} \mapsto f_{i+1,\ell}$  for  $\ell \leq i < i'$ . Now we have that  $f_{\ell,\ell} \neq 0$ , and each nonzero entry below and to the right of it is of equal or higher degree. Using  $f_{\ell,\ell}$  as a pivot entry, we perform elementary row and column operations on  $\varphi$  to eliminate all entries below and to the right, so that  $f_{i,\ell} = f_{\ell,j} = 0$  for all  $i, j > \ell$ . Notice that the homogeneity of  $\varphi$  guarantees that at each step, the property in (2.3.1.1) is preserved. Continuing this process,  $\varphi$  is reduced to an  $t \times s$  matrix of the form

$$\left[ \begin{array}{ccc} f_{1,1} & & 0 \\ & \ddots & \\ 0 & & f_{s,s} \\ \hline & & 0 \end{array} \right]$$

where, for each  $1 \leq i \leq s$ ,  $f_{i,i} = \alpha_i x^{c_i - r_i}$  for some  $\alpha_i \in k$  and some positive integers  $c_i, r_i$ . Letting  $h_i = (\alpha_i)^{-1} f_{i,i}$ , the result follows.  $\square$

**Theorem 2.3.2.** *Every semifree DG module of finite rank over  $A = k[x]$  is in the image of  $\text{Tot} : \mathbf{DGr}(A) \rightarrow \mathbf{DDG}(A)$ .*

*Proof.* Let  $M$  be a rank  $d$  semifree DG module over  $A = k[x]$ . By Lemma 2.3.1, we can assume the homology of  $M$  to have the form

$$(2.3.2.1) \quad H(M) \cong \bigoplus_{i=1}^s \frac{\Sigma^{r_i} A}{h_i \Sigma^{c_i} A} \oplus \bigoplus_{i=s+1}^t \Sigma^{r_i} A$$

for some  $1 \leq t \leq d$  and  $0 \leq s \leq t$ , and positive integers  $r_i, c_j$ . (Of course,  $s = 0$  corresponds to the case that  $H(M)$  is a free  $A$ -module.) Consider the deleted minimal graded free resolution  $F$  of  $H(M)$  given by the following:

$$F : \begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^{c_1} A & \xrightarrow{h_1} & \Sigma^{r_1} A & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ & & \vdots & & \vdots & & \\ & & \oplus & & \oplus & & \\ 0 & \longrightarrow & \Sigma^{c_s} A & \xrightarrow{h_s} & \Sigma^{r_s} A & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ & & 0 & \longrightarrow & \Sigma^{r_{s+1}} A & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ & & \vdots & & \vdots & & \\ & & \oplus & & \oplus & & \\ & & 0 & \longrightarrow & \Sigma^{r_t} A & \longrightarrow & 0 \end{array}$$

Notice that this resolution is precisely the one given in the statement of Lemma 2.3.1.

To complete the proof, we will show that  $\text{Tot } F \simeq M$ . While it is clear that  $H(\text{Tot } F) \cong H(M)$ , it remains to exhibit a chain map (or sequence thereof) which induces this isomorphism. To this end, for each  $1 \leq i \leq t$ , let  $G_i$  be the subcomplex which is given by the  $i^{\text{th}}$  summand of  $F$ . Namely:

$$G_i : \quad \begin{cases} 0 \rightarrow \Sigma^{c_i} A \xrightarrow{h_i} \Sigma^{r_i} A \rightarrow 0 & 1 \leq i \leq s \\ 0 \rightarrow \Sigma^{r_i} A \rightarrow 0 & s < i \leq t \end{cases}$$

Our goal is to define a family of chain maps  $\mu^i : \text{Tot } G_i \rightarrow M$  such that the chain map  $\mu : \text{Tot } F \rightarrow M$  given by

$$(2.3.2.2) \quad \mu = (\mu^i) : \bigoplus_{i=1}^t \text{Tot } G_i \rightarrow M$$

induces an isomorphism in homology. In particular, for each  $1 \leq i \leq t$ , we define  $\mu^i$  by a family  $\mu_j^i : (\text{Tot } G_i)_j \rightarrow M_j$  of homomorphisms of vector spaces over  $k$  in such a way that each  $\mu^i$  is  $A$ -linear and furthermore commutes with the differentials of  $\text{Tot } G_i$  and  $M$ .

In order for the reader to better understand the action of  $\mu^i$  on  $\text{Tot } G_i$ , we include the following diagram for  $1 \leq i \leq s$ , wherein complexes are written vertically. Note that the diagonal maps represent the nontrivial component of  $\partial^{\text{Tot } G_i}$ .

$$\begin{array}{ccccc}
& \vdots & & \vdots & \vdots \\
& \oplus & & \oplus & \downarrow \\
& (\Sigma^{c_i+1}A)_{c_i+2} \oplus (\Sigma^{r_i}A)_{c_i+2} & \xrightarrow{\mu_{c_i+2}^i} & M_{c_i+2} & \downarrow \partial_{c_i+2}^M \\
& \oplus & \searrow h_i & \oplus & \\
& (\Sigma^{c_i+1}A)_{c_i+1} \oplus (\Sigma^{r_i}A)_{c_i+1} & \xrightarrow{\mu_{c_i+1}^i} & M_{c_i+1} & \downarrow \partial_{c_i+1}^M \\
& \oplus & \searrow h_i & \oplus & \\
& 0 \oplus (\Sigma^{r_i}A)_{c_i} & \xrightarrow{\mu_{c_i}^i} & M_{c_i} & \downarrow \partial_{c_i}^M \\
(2.3.2.3) & \oplus & & \oplus & \downarrow \partial_{c_i}^M \\
& \vdots & & \vdots & \vdots \\
& & & \oplus & \downarrow \partial_{r_i-1}^M \\
& & & (\Sigma^{r_i}A)_{r_i} & \xrightarrow{\mu_{r_i}^i} M_{r_i} \\
& & & \oplus & \downarrow \partial_{r_i}^M \\
& & & 0 & \xrightarrow{\mu_{r_i-1}^i} M_{r_i-1} \\
& & & \oplus & \downarrow \\
& & & \vdots & \vdots
\end{array}$$

Although this diagram only depicts the chain map for  $1 \leq i \leq s$ , the situation is straightforward for larger values of  $i$ ; certainly, these cases represent the torsion-free part of  $H(M)$ . Thus, for  $s < i \leq t$ , we have the following:

$$\mu_j^i : \begin{cases} 0 \rightarrow M_j & j < r_i \\ (\Sigma^{r_i}A)_j \rightarrow M_j & j \geq r_i \end{cases}$$

Now consider the following isomorphism of graded  $A$ -modules:

$$\varphi : \bigoplus_{i=1}^s \frac{\Sigma^{r_i}A}{h_i \Sigma^{c_i}A} \oplus \bigoplus_{i=s+1}^t \Sigma^{r_i}A \rightarrow H(M)$$

Fix  $1 \leq i \leq t$  and let  $0 \neq z_i \in M_{r_i}$  be a cycle defined such that  $\varphi(\sigma^{r_i}1) = \text{cls}(z_i) \in H_{r_i}(M)$ . Since  $\varphi$  is  $A$ -linear,  $\varphi(x^\ell \sigma^{r_i}1) = \text{cls}(x^\ell z_i)$  for each nonnegative integer  $\ell$ . Notice that, for small enough values of  $\ell$ , these classes are nonzero. To be precise,  $\text{cls}(x^\ell z_i) = 0$  if and only if  $1 \leq i \leq s$  and  $\ell \geq c_i - r_i$ . Therefore, for each  $1 \leq i \leq s$ , there exists  $m_i \in M_{c_i+1}$  such that  $\partial_{c_i+1}^M(m_i) = x^{c_i-r_i} z_i$ .

We shall now proceed to define, for each  $i$  and  $j$ , a basis  $U_j^i$  for  $(\text{Tot } G_i)_j$  over  $k$ .

$$U_j^i \supseteq \begin{cases} \{x^{j-r_i} \sigma^{r_i} 1, (-1)^{j-c_i-1} x^{j-c_i-1} \sigma^{c_i+1} 1\} & 1 \leq i \leq s \text{ and } j > c_i \\ \{x^{j-r_i} \sigma^{r_i} 1\} & \text{otherwise} \end{cases}$$

Finally, we define the map  $\mu_j^i : (\text{Tot } G_i)_j \rightarrow M_j$  in the following way:

$$\mu_j^i(u) = \begin{cases} x^{j-r_i} z_i & u = x^{j-r_i} \sigma^{r_i} 1 \\ x^{j-c_i-1} m_i & u = (-1)^{j-c_i-1} x^{j-c_i-1} \sigma^{c_i+1} 1 \\ 0 & \text{otherwise} \end{cases}$$

By construction,  $\mu^i = (\mu_j^i)$  is an  $A$ -linear degree zero chain map between  $\text{Tot } G_i$  and  $M$ . Further, one can easily check that the Leibniz rule is satisfied, so that the map  $\mu : \text{Tot } F \rightarrow M$  given in (2.3.2.2) is a morphism of DG modules. The above construction also guarantees that  $\mu$  establishes a one-to-one correspondence between the generators of homology of  $\text{Tot } F = \bigoplus_{i=1}^m \text{Tot } G_i$  and that of  $M$ . The result follows.  $\square$

Now we will illustrate the practical use of the theorem with an example.

**Example 2.3.3.** Let  $M$  be the rank five semifree DG module over  $A = k[x]$  with a basis  $\{e_1, e_2, e_3, e_4, e_5\}$  such that  $|e_1| = 0$ ,  $|e_2| = 2$ ,  $|e_3| = 4$ ,  $|e_4| = 8$ ,  $|e_5| = 9$ , and where differential of  $M$  is given by the following.

$$\begin{aligned} \partial(e_1) &= 0 \\ \partial(e_2) &= 0 \\ \partial(e_3) &= 0 \\ \partial(e_4) &= x^7 e_1 + x^5 e_2 \\ \partial(e_5) &= x^4 e_3 \end{aligned}$$

We obtain the following decomposition in homology

$$\begin{aligned} \text{H}(M) &= \frac{Ae_1 \oplus Ae_2 \oplus Ae_3}{A(x^7 e_1 + x^5 e_2) \oplus Ax^4 e_3} \\ (2.3.3.1) \quad &\cong \frac{A(x^2 e_1 + e_2)}{A(x^7 e_1 + x^5 e_2)} \oplus \frac{Ae_3}{Ax^4 e_3} \oplus Ae_1 \end{aligned}$$

whence we obtain a deleted minimal free resolution  $F$  of  $\text{H}(M)$ , given by:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^7 A & \xrightarrow{x^5} & \Sigma^2 A & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ 0 & \longrightarrow & \Sigma^8 A & \xrightarrow{x^4} & \Sigma^4 A & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ & & 0 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

We will utilize the proof of Theorem 2.3.2 to show that  $\text{Tot } F \simeq M$ . From  $F$  we obtain the subcomplexes:

$$\begin{aligned} G_1 : \quad & 0 \rightarrow \Sigma^7 A \xrightarrow{x^5} \Sigma^2 A \rightarrow 0 \\ G_2 : \quad & 0 \rightarrow \Sigma^8 A \xrightarrow{x^4} \Sigma^4 A \rightarrow 0 \\ G_3 : \quad & 0 \rightarrow A \rightarrow 0 \end{aligned}$$

Now, referring to the decomposition of homology in (2.3.3.1), we have that the cycles generating  $\text{H}(M)$  over  $A$  are  $z_1 = x^2 e_1 + e_2$ ,  $z_2 = e_3$ , and  $z_3 = e_1$ . Clearly, for each  $j \in \mathbb{N}$ , the basis  $U_j^1$  of  $(\text{Tot } G_3)_j$  over  $k$  must be chosen to contain  $\{x^j \sigma^0\}$ .

Furthermore, the respective bases of  $(\text{Tot } G_1)_j$  and  $(\text{Tot } G_2)_j$  over  $k$  are given as follows.

$$U_j^1 \supseteq \begin{cases} \{x^{j-2}\sigma^2 1\} & j < 8 \\ \{x^{j-2}\sigma^2 1, (-1)^{j-8}x^{j-8}\sigma^8 1\} & j \geq 8 \end{cases}$$

$$U_j^2 \supseteq \begin{cases} \{x^{j-4}\sigma^4 1\} & j < 9 \\ \{x^{j-4}\sigma^4 1, (-1)^{j-9}x^{j-9}\sigma^9 1\} & j \geq 9 \end{cases}$$

We can now exploit these bases in order to define chain maps  $\mu^i = (\mu_j^i) : (\text{Tot } G_i)_j \rightarrow M_j$ . We have

$$\mu_j^1(u) = \begin{cases} x^j e_1 + x^{j-2} e_2 & u = x^{j-2} \sigma^2 1 \\ x^{j-8} e_4 & u = (-1)^{j-8} x^{j-8} \sigma^8 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_j^2(u) = \begin{cases} x^{j-4} e_3 & u = x^{j-4} \sigma^4 1 \\ x^{j-9} e_5 & u = (-1)^{j-9} x^{j-9} \sigma^9 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_j^3(u) = \begin{cases} x^j e_1 & u = x^j \sigma^0 1 \\ 0 & \text{otherwise} \end{cases}$$

so that  $\mu : \text{Tot } F \rightarrow M$  is given by  $\mu(x) = (\mu^1(x), \mu^2(x), \mu^3(x))$  for each  $x \in \text{Tot } F$ .

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